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The Holstein–Primakoff and Dyson realizations for the Lie superalgebra *gl*(*m*/*n* + 1)

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Abstract. The known Holstein–Primakoff and Dyson realizations for $gl(n+1)$, $n = 1, 2, \ldots$, in terms of Bose operators (Okubo S 1975 *J. Math. Phys.* **16** 528) are generalized to the class of the Lie superalgebras $gl(m/n+1)$ for any n and m. Formally the expressions are the same as for $gl(m+n+1)$, however, both Bose and Fermi operators are involved.

Recently an analogue of the Dyson (D) and of the Holstein–Primakoff (H–P) realization for all Lie superalgebras $sl(1/n)$ [1] was given. In the present paper the results are extended to the case of the Lie superalgebras $gl(m/n+1)$ for any m and n.

Initially the H–P and the D realizations were given for $sl(2)$ [2, 3]. The generalization for $gl(n)$ is due to Okubo [4]. The extension to the case of quantum algebras is available so far only for $sl(2)$ [5] and $sl(3)$ [6]. To the best of our knowledge, apart from [1] other results on H–P or D realizations for Lie superalgebras have not been published in the literature so far.

The motivation in the present work stems from the various applications of the Holstein–Primakoff and of the Dyson realizations in theoretical physics. Beginning with [2] and [3] the H–P and D realizations were constantly used in condensed matter physics. Some other early applications can be found in the book of Kittel [7] (more recent results are contained in [8]). For applications in nuclear physics see [9, 10] and the references therein, but there are, certainly, several other publications. In view of the importance of the Lie superalgebras for physics, one could expect that extensions of the Dyson and of the Holstein–Primakoff realizations to $\mathbf{Z}_2$-graded algebras may be of interest too.

We recall the H–P realization of $gl(n+1)$. The Weyl generators E_{AB}, $A, B = 1, \ldots, n+1$, of $gl(n+1)$ satisfy the commutation relations:

$$[E_{AB}, E_{CD}] = \delta_{BC} E_{AD} - \delta_{AD} E_{CD}. \tag{1}$$

Let $b_i^{\pm}$, $n = 1, \ldots, n$, be n pairs of Bose creation and annihilation operators (CAOs),

$$[b_i^-, b_j^+] = \delta_{ij} \qquad [b_i^+, b_j^+] = [b_i^-, b_j^-] = 0 \qquad i, j = 1, \ldots, n. \tag{2}$$

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Then for any non-negative integer p, $p \in \mathbf{Z}_+$, the H–P realization π of $gl(n+1)$ is defined on the generators as follows [4]:

$$\pi(E_{ij}) = b_i^+ b_j^- \qquad i, j = 1, \ldots, n \tag{3}$$

$$\pi(E_{i,n+1}) = b_i^+ \sqrt{p - \sum_{k=1}^n b_k^+ b_k^-} \pi(E_{n+1,i}) = \sqrt{p - \sum_{k=1}^n b_k^+ b_k^-} b_i^-$$

$$\pi(E_{n+1,n+1}) = p - \sum_{k=1}^n b_k^+ b_k^-. \tag{4}$$

($3a$) only gives the known Jordan–Schwinger (J–S) realization of $gl(n)$ in terms of n pairs of Bose CAOs. Therefore, the H–P (and also the D) realizations allow one to express the higher rank algebra $gl(n+1)$ also through n pairs of Bose CAOs.

Let us fix some notation. Unless otherwise stated $A, B, C, D = 1, 2, \ldots, m+n+1$ and $i, j, k, l \in \{1, 2, \ldots, m+n = M\} \equiv \mathbf{M}$; $[x, y] = xy - yx$, $\{x, y\} = xy + yx$; $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$; $\langle A \rangle = \bar{1}$, if $A \leqslant m$; $\langle A \rangle = \bar{0}$, if $A > m$.

We proceed to define $gl(m/n+1)$ in a representation independent way [11]. Let U be the (free complex) associative unital (= with unity) algebra of the indeterminants $\{E_{AB} | A, B = 1, \ldots, M+1\}$ subject to the relations

$$E_{AB}E_{CD} - (-1)^{(\langle A \rangle + \langle B \rangle)(\langle C \rangle + \langle D \rangle)} E_{CD}E_{AB} = \delta_{BC}E_{AD} - (-1)^{(\langle A \rangle + \langle B \rangle)(\langle C \rangle + \langle D \rangle)} E_{CB}. \tag{5}$$

Introduce a $\mathbf{Z}_2$-grading on U, induced from

$$\deg(E_{AB}) = \langle A \rangle + \langle B \rangle. \tag{6}$$

Then U is an (infinite-dimensional) associative superalgebra, which is also a Lie superalgebra (LS) with respect to the supercommutator $[\![\,,]\!]$ defined between every two homogeneous elements $x, y \in U$ as

$$[\![x, y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx. \tag{7}$$

Its finite-dimensional subspace

$$\text{lin.env.}\{E_{AB}, [\![E_{AB}, E_{CD}]\!] | A, B, C, D = 1, \ldots, m+n+1\} \subset U \tag{8}$$

gives the Lie superalgebra $gl(m/n+1)$; $U = U[gl(m/n+1)]$ is its universal enveloping algebra. The relations (4) are the supercommutation relations on $gl(m/n+1)$:

$$[\![E_{AB}, E_{CD}]\!] = \delta_{BC}E_{AD} - (-1)^{(\langle A \rangle + \langle B \rangle)(\langle C \rangle + \langle D \rangle)} E_{CB}. \tag{9}$$

One can certainly define $gl(m/n+1)$ in its matrix representation. In that case E_{AB} is a $(m+n+1) \times (m+n+1)$ matrix with one on the intersection of the Ath row and Bth column and zero elsewhere.

The Dyson and the Holstein–Primakoff realizations are different embeddings of $gl(m/n+1)$ into the algebra $W(m/n)$ of all polynomials of m pairs of Fermi CAOs and n pairs of Bose CAOs. The precise definition of $W(m/n)$ is the following. Let $A_i^\pm, i \in \mathbf{M}$ be $\mathbf{Z}_2$-graded indeterminates:

$$\deg(A_i^\pm) = \langle i \rangle. \tag{10}$$

Then $W(m/n)$ is the associative unital superalgebra of all $A_i^\pm$, subject to the relations

$$[\![A_i^-, A_j^+]\!] = \delta_{ij} \qquad [\![A_i^+, A_j^+]\!] = [\![A_i^-, A_j^-]\!] = 0. \tag{11}$$

With respect to the supercommutator (7) $W(m/n)$ is also a Lie superalgebra.

From (11) one concludes that $A_1^\pm, \ldots, A_m^\pm$ are Fermi CAOs, which are odd variables; $A_{m+1}^\pm, \ldots, A_{m+n}^\pm$ are Bose CAOs, which are even. The Bose operators commute with the Fermi operators.

Proposition 1 (Dyson realization). The linear map $\varphi : gl(m/n+1) \to W(m/n)$, defined on the generators as

$$\varphi(E_{ij}) = A_i^+ A_j^- \qquad i, j = 1, \ldots, M \tag{12a}$$

$$\varphi(E_{i,M+1}) = A_i^+ \qquad \varphi(E_{M+1,i}) = \left(p - \sum_{k=1}^{M} A_k^+ A_k^-\right) A_i^-$$

$$\varphi(E_{M+1,M+1}) = p - \sum_{k=1}^{M} A_k^+ A_k^- \tag{12b}$$

is an isomorphism of $gl(m/n+1)$ into $W(m/n)$ for any number p.

Proof. The images $\varphi(E_{AB})$ are linearly independent in $W(m/n)$. It is straightforward to verify that they preserve the supercommutation relations (9),

$$[\![\varphi(E_{AB}), \varphi(E_{CD})]\!] = \delta_{BC}\varphi(E_{AD}) - (-1)^{(\langle A\rangle + \langle B\rangle)(\langle C\rangle + \langle D\rangle)}\varphi(E_{CB}). \tag{13}$$

In the intermediate computations the following relation is useful

$$[N, A_i^\pm] = \pm A_i^\pm \qquad \text{where} \qquad N = \sum_{k=1}^{M} A_k^+ A_k^-. \tag{14}$$

The Dyson realization defines an infinite-dimensional representation of $gl(m/n+1)$ (for $m > 0$) in the Fock space $F(m/n)$ with orthonormed basis

$$|K) \equiv |k_1, \ldots, k_M) = \frac{(A_1^+)^{k_1} \ldots (A_M^+)^{k_M}}{\sqrt{k_1! \ldots k_M!}} |0\rangle \qquad k_1, \ldots, k_m = 0, 1;$$
$$k_{m+1}, \ldots, k_M \in \mathbf{Z}_+. \tag{15}$$

Let $|K)_{\pm i}$ (respectively $|K)_{i,-j}$) be a vector obtained from $|K)$ after a replacement of k_i with $k_i \pm 1$ (respectively $k_i \to k_i + 1, k_j \to k_j - 1$). The transformations of the basis (15) under the action of the CAOs read

$$A_i^+|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\sqrt{1 + (-1)^{\langle i\rangle}k_i}\,|K)_i$$
$$A_i^-|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\sqrt{k_i}\,|K)_{-i}. \tag{16}$$

As a consequence one obtains the transformations of the $gl(m/n+1)$ module $F(m/n)$

$$\varphi(E_{i,M+1})|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\sqrt{1 + (-1)^{\langle i\rangle}k_i}\,|K)_i \tag{17a}$$

$$\varphi(E_{M+1,i})|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\left(p + 1 - \sum_{j=1}^{M} k_j\right)\sqrt{k_i}\,|K)_{-i} \tag{17b}$$

$$\varphi(E_{M+1,M+1})|K) = \left(p + 1 - \sum_{j=1}^{M} k_j\right)|K) \tag{17c}$$

$$\varphi(E_{ii})|K) = k_i|K) \tag{17d}$$

$$\varphi(E_{ij})|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})+\langle j\rangle(k_1+\cdots+k_{j-1})}\sqrt{k_j(1 + (-1)^{\langle i\rangle}k_i)}\,|K)_{-j,i} \qquad i < j \tag{17e}$$

$$\varphi(E_{ij})|K) = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1}+1)+\langle j\rangle(k_1+\cdots+k_{j-1})}\sqrt{k_j(1 + (-1)^{\langle i\rangle}k_i)}\,|K)_{-j,i} \qquad i > j. \tag{17f}$$

If p is not a positive integer, $p \notin \mathbf{N}$, $F(m/n)$ is a simple $gl(m/n+1)$ module. For any positive integer p, $p \in \mathbf{N}$, the representation of $gl(m/n+1)$ in $F(m/n)$ is indecomposable. The subspace

$$F(p; m/n)_{\text{inv}} = \text{lin.env.}\{|K)|k_1 + \cdots + k_M > p\} \subset F(m/n) \tag{18}$$

is an infinite-dimensional subspace, invariant with respect to $gl(m/n+1)$. The factor spaces

$$F(p;m/n)_0 \equiv F(m/n)/F(p;m/n)_{\rm inv} = \text{lin.env.}\{|K)|p \geqslant k_1+\cdots+k_M\} \qquad p=1,2,\ldots \tag{19}$$

are finite-dimensional irreducible $gl(m/n+1)$ modules.

The advantage of the Dyson realization (12) is its simplicity. Its disadvantage is that the Fock representation of $gl(m/n+1)$ is not unitarizable. The latter, the representation to be unitarizable, is usually required for physical reasons. We recall that a representation φ of a (super)algebra L in a Hilbert space V is unitarizable with respect to an antilinear anti-involution $\omega : L \to L$ and a scalar product (,) in V, if

$$(\varphi(a)x,y) = (x,\varphi(\omega(a))y) \qquad \forall a \in L, \quad \forall x,y \in V. \tag{20}$$

The Dyson representation in $F(m/n)$ is not unitarizable with respect to the 'compact' anti-involution

$$\omega(E_{AB}) = E_{BA} \qquad A,B=1,\ldots,M+1. \tag{21}$$

The factor modules $F_0(p;m/n)$, $p \in \mathbf{N}$, however do carry unitarizable representations for any $p \in \mathbf{N}$. In order to show this it is convenient to introduce a new basis within each $F_0(p;m/n)$, which we postulate to be orthonormed

$$|K\rangle = \sqrt{\left(p-\sum_{j=1}^{M}k_j\right)!}|K). \tag{22}$$

In this basis the transformation relations (17) read

$$\varphi(E_{i,M+1})|K\rangle = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\sqrt{(1+(-1)^{\langle i\rangle}k_i)\left(p-\sum_{j=1}^{M}k_j\right)}|K\rangle_i \tag{23a}$$

$$\varphi(E_{M+1,i})|K\rangle = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})}\sqrt{k_i\left(p+1-\sum_{j=1}^{M}k_j\right)}|K\rangle_{-i} \tag{23b}$$

$$\varphi(E_{M+1,M+1})|K\rangle = \left(p+1-\sum_{j=1}^{M}k_j\right)|K\rangle \tag{23c}$$

$$\varphi(E_{ii})|K\rangle = k_i|K\rangle \tag{23d}$$

$$\varphi(E_{ij})|K\rangle = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1})+\langle j\rangle(k_1+\cdots+k_{j-1})}\sqrt{k_j(1+(-1)^{\langle i\rangle}k_i)}|K\rangle_{-j,i} \qquad i<j \tag{23e}$$

$$\varphi(E_{ij})|K\rangle = (-1)^{\langle i\rangle(k_1+\cdots+k_{i-1}+1)+\langle j\rangle(k_1+\cdots+k_{j-1})}\sqrt{k_j(1+(-1)^{\langle i\rangle}k_i)}|K\rangle_{-j,i} \qquad i>j. \tag{23f}$$

It is straightforward to check that (20) holds with respect to the anti-involution (21). Hence the representation of $gl(m/n+1)$ is unitarizable within every space $F_0(p;m/n)$, $p \in \mathbf{N}$. The next proposition is closely related to the result we have just obtained. □

Proposition 2 (Holstein–Primakoff realization). The linear map $\pi : gl(m/n+1) \to W(m/n)$, defined on the generators as

$$\pi(E_{ij}) = A_i^+A_j^- \qquad i,j=1,\ldots,M \tag{24a}$$

$$\pi(E_{i,M+1}) = A_i^+ \sqrt{p - \sum_{j=1}^{M} A_j^+ A_j^-} \qquad \pi(E_{M+1,i}) = \sqrt{p - \sum_{k=1}^{M} A_k^+ A_k^-} A_i^-$$

$$\pi(E_{M+1,M+1}) = p - \sum_{k=1}^{M} A_k^+ A_k^- \tag{24b}$$

is an isomorphism of $gl(m/n+1)$ into $W(m/n)$ for any positive integer p.

Proof. Acting with the $gl(m/n+1)$ generators on the basis (15) one obtains the same transformation relations (23) with the only difference that everywhere in (23), $|K\rangle$ have to be replaced with $|K)$. The proof can be carried out also purely algebraically, using the supercommutation relations (11). To this end the following formula is useful

$$f(N)A_i^{\pm} = A_i^{\pm} f(N \pm 1) \qquad N = \sum_{j=1}^{M} A_j^+ A_j^- \tag{25}$$

where $f(z)$ is any (analytical) function in z.

The representation π is defined in the entire Fock space. Observe that with respect to $\pi(E_{AB}), A, B = 1, \ldots, M+1$, the Fock space resolves into a direct sum of two invariant (and moreover irreducible) subspaces (which was not the case with the Dyson representation):

$$F(p; m/n)_0 = \text{lin.env.}\{|K)|p \geqslant k_1 + \cdots + k_M\}$$
$$F(p; m/n)_{\text{inv}} = \text{lin.env.}\{|K)|k_1 + \cdots + k_M > p\}. \tag{26}$$

This property is due to the factors $\sqrt{p - \sum_{j=1}^{M} k_j}$ and $\sqrt{p + 1 - \sum_{j=1}^{M} k_j}$ in (23*a*) and (23*b*), respectively.

In the case $m = 0$ the Holstein–Primakoff realization (24) reduces to the Holstein–Primakoff realization (3) of $gl(n+1)$ in terms of only Bose operators. Replacing in (24) all $A_i^{\pm}$ with Bose CAOs, one obtains the H–P realization of $gl(m+n+1)$. The case $n = 0$ yields a Fermi realization of the Lie superalgebras $gl(m/1)$. Its restriction to $sl(m/1)$ coincides with the results announced in [1]. □

Let us note in conclusion that explicit expressions for all finite-dimensional irreducible representations of $gl(m/1)$ and a large class of representations of $gl(m/n+1)$ are available [12]. They have been generalized also to the quantum case [13]. The formulae are, however, extremely involved. The Dyson and the Holstein–Primakoff representations lead to a small part of all representations. Their description is, however, simple and it is realized in familiar for physics Fock spaces.

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